Solutions for Exam Physics Laboratory 1: Data and error analysis
10 November 2016

Exercise 1

(4 points total)

a) $\Delta p = 0.3$ kPa so $p = 2101.3 \pm 0.3$ kPa $= (2101.3 \pm 0.3) \cdot 10^3$ Pa
(1 point)

b) $\Delta\lambda = 0.08$ nm so $\lambda = 589.63 \pm 0.08$ nm $= 0.58963 \pm 0.00008$ μm
(1 point)

c) $\Delta T = 0.09$ mK so $T = 4187.00 \pm 0.09$ mK $= 4.18700 \pm 0.00009$ K $= (4.18700 \pm 0.00009) \cdot 10^6$ μK (the relatively small error warrants more significant digits than originally given)
(1 point)

d) $\Delta R = 0.3$ MΩ so $R = 5.6 \pm 0.3$ MΩ $= (5.6 \pm 0.3) \cdot 10^3$ kΩ
(1 point)

Exercise 2

(10 points total)

a) $c = \sqrt{K/\rho} \Leftrightarrow c^2 = K/\rho \Leftrightarrow K = \rho c^2$ so (SI) units of K are $[K] = [\rho][c]^2 =$ (kg m^{-3}) (m^2 s^{-2}) $=$ kg m^{-1} s^{-2}.
(1 point)

b) $L = 10.00$ m and $t_1 = 6.85$ s $\Rightarrow c = \frac{L}{t_1} = 10.00/6.85 = 1.4599$ m s^{-1}. *(rounding to 1.46 allowed, but not necessary) (subtract 0.5 point if units are not given)*
(1 point)

Note: the literature value is 1460 m s^{-1}, so 1000× higher. I should have written $t_1 = 6.85$ ms and not $t_1 = 6.85$ s (my mistake).

c) $c = \frac{L}{t_1} \Rightarrow \left(\frac{\Delta c}{c}\right)^2 = \left(\frac{\Delta L}{L}\right)^2 + \left(\frac{\Delta t_1}{t_1}\right)^2 = (0.01/10.00)^2 + (1\%)^2 = 1 \cdot 10^{-6} + 1 \cdot 10^{-4} = 1.01 \cdot 10^{-4} \Rightarrow \frac{\Delta c}{c} = 0.01005 \approx 0.01 = 1\%$. *(1 point for correct formula, 1 point for correct value)*
(2 points total)

d) $K = \rho c^2 = 13.5 \cdot 10^3 \times 1.4599^2 = 28773$ kg m^{-1} s^{-2}. *(rounding not yet necessary) (subtract 0.5 point if units are not given)*
(1 point)

If substitute answer $c = 2580$ m s^{-1} is used, then $K = 13.5 \cdot 10^3 \times 2580^2 = 8.9861 \cdot 10^{10}$ kg m^{-1} s^{-2}.

e)

$$K = \rho c^2 \Rightarrow \left(\frac{\Delta K}{K}\right)^2 = \left(\frac{\Delta\rho}{\rho}\right)^2 + \left(\frac{\Delta c^2}{c^2}\right)^2 = \left(\frac{\Delta\rho}{\rho}\right)^2 + \left(2\frac{\Delta c}{c}\right)^2 = \left(\frac{\Delta\rho}{\rho}\right)^2 + 4\left(\frac{\Delta c}{c}\right)^2$$

so

$$\frac{\Delta K}{K} = \sqrt{\left(\frac{\Delta\rho}{\rho}\right)^2 + 4\left(\frac{\Delta c}{c}\right)^2},$$

or with partial derivatives:

$$(\Delta K)^2 = \left(\frac{\partial K}{\partial\rho}\Delta\rho\right)^2 + \left(\frac{\partial K}{\partial c}\Delta c\right)^2 = (c^2\Delta\rho)^2 + (2\rho c\Delta c)^2 = (c^2\Delta\rho)^2 + 4(\rho c\Delta c)^2$$

so

$$\Delta K = \sqrt{(c^2\Delta\rho)^2 + 4\,(\rho c\Delta c)^2}$$

Both methods are equivalent of course, only one of these methods needs to be used, therefore only one correct formula needs to be given, either for the relative error or for the absolute error. *(2 points for correct formula)*

Putting in numbers: $\frac{\Delta K}{K} = \sqrt{\left(\frac{0.4}{13.5}\right)^2 + 4(0.01)^2} = 0.03575 \approx 0.04 = 4\%$ from which follows $\Delta K = \frac{\Delta K}{K}K = 0.03575 \cdot 28773 = 1028.6 \approx 2 \cdot 10^3$ kg m^{-1} s^{-2}. *(round upward) (1 point for relative error and 1 point for absolute error; subtract 0.5 point if units are not given for absolute error)(subtract 0.5 point if incorrect number of significant digits or if not rounded)*

If substitute answer $c = 2580 \pm 13$ m s^{-1} is used, then $\frac{\Delta K}{K} = 0.03130 \approx 0.04 = 4\%$ and $\Delta K = 2.8123 \cdot 10^9 \approx 3 \cdot 10^9$ kg m^{-1} s^{-2}.
(total 4 points)

f) $K = (29 \pm 2) \cdot 10^3$ kg m^{-1} s^{-2}.

If substitute answer $c = 2580 \pm 13$ m s^{-1} is used, then $K = (9.0 \pm 0.3) \cdot 10^{10}$ kg m^{-1} s^{-2}.
(1 point: only if everything correct, otherwise 0 points)

Exercise 3

(4 points total)

a) $n_1 = 2.417 \pm 0.001$, $n_2 = 2.415 \pm 0.003$, $n_3 = 2.421 \pm 0.005$ (not just all n_1) so $w_1 = s_1^{-2} = 1.0000 \cdot 10^6$, $w_2 = s_2^{-2} = 0.1111 \cdot 10^6$, $w_3 = s_3^{-2} = 0.0400 \cdot 10^6$
$n_{avg} = \frac{\sum w_i n_i}{\sum w_i} = 2.4169 \approx 2.417$ *(1 point for correct weights + 1 point for correct weighted average)*
(2 points total)

b) $\frac{1}{s_{n_{avg}}}^2 = \sum w_i = 1.1511 \cdot 10^6 \Rightarrow s_{n_{avg}} = (1.1511 \cdot 10^6)^{-1/2} = 9.3205 \cdot 10^{-4} \approx 0.001$ (round error upward) In the correct notation: $n_{avg} \pm \Delta n = 2.417 \pm 0.001$
(2 points total)

Exercise 4

(9 points total)

a) Probability $P = f(k) = \frac{\lambda^k}{k!}e^{-\lambda}$ with $\lambda = 3.0$ and $k = 3$ so $P = \frac{3^3}{3!}e^{-3} = 0.224$ so $P = 22\%$ so answer **C**. Any calculation or argument may be ignored, but if a student starts OK but makes a small mistake later on, then 0.5 point may be given.
(1 point)

b) $\rho_1 = 9.73$, $\rho_2 = 9.68$, $\rho_3 = 9.77$, $\rho_4 = 9.72$, $\rho_5 = 9.70$ kg m^{-3} so $\rho_{av} = 9.72$. Subtract 1 point if incorrect number of significant digits Subtract 0.5 point if units are not given
(2 points)

c) $N = 5; s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{5}(\rho_i - \rho_{av})^2} =$
$\sqrt{\frac{1}{4}((0.01)^2 + (-0.04)^2 + (0.05)^2 + (0.00)^2 + (-0.02)^2)} = \sqrt{0.0046/4} = \sqrt{0.00115} =$ $0.0339 \approx 0.04$ kg m^{-3}. (round error upward) Subtract 0.5 point if units are not given
(2 points)

d) $s_m = s/\sqrt{N} = 0.0339/\sqrt{5} = 0.0152 \approx 0.02$ kg m^{-3}. (round error upward). If $s_m = s/\sqrt{N} = 0.04/\sqrt{5} = 0.0179 \approx 0.02$ kg m^{-3} is given, this is also counted as correct. Subtract 0.5 point if units are not given
(2 points)

e) $z_1 = (\rho_1 - \rho_{av})/\sigma = (9.710 - 9.730)/0.025 = -0.02/0.025 = -0.8$ and $z_2 = (\rho_2 - \rho_{av})/\sigma = (9.770 - 9.730)/0.025 = 0.04/0.025 = 1.6$ so probability $P = \int_{z_1}^{z_2} N(y)dy = \int_{-0.8}^{1.6} N(y)dy = \int_0^{0.8} N(y)dy + \int_0^{1.6} N(y)dy \approx 0.2881 + 0.4452 = 0.7333$ so $P \approx 73\%$. *(1 point for correct method, 1 point for correct numbers)*
(2 points total)

Exercise 5

(13 points total)

x	$y \pm \Delta y$	r
-3	6 ± 1	-0.1379
-1	4 ± 2	0.6552
2	-2 ± 2	-1.1552
4	-3 ± 1	0.6379

Table 1: Observations for exercise 5, now with residuals.

$$a = \frac{N\sum x_i y_i - \sum x_i \sum y_i}{N\sum x_i^2 - \left(\sum x_i\right)^2},$$

$$(\Delta a)^2 = \left(\frac{1}{\sum x_i^2 - N\overline{x}^2}\right)\frac{\sum r_i^2}{N-2},$$

$$(\Delta b)^2 = \left(\frac{1}{N} + \frac{\overline{x}^2}{\sum x_i^2 - N\overline{x}^2}\right) \frac{\sum r_i^2}{N-2}.$$

a) $N = 4, \sum x_i = 2, \sum y_i = 5, \sum x_i^2 = 30, \sum x_i y_i = -38$
$\Rightarrow a = (4 \cdot -38 - 2 \cdot 5)/(4 \cdot 30 - \{2\}^2) = -1.3966 \approx -1.4$
and $\overline{x} = 0.50, \overline{y} = 1.25 \Rightarrow b = \overline{y} - a\overline{x} = 1.9483 \approx 1.9.$
(2 points total)

b) $\sum r_i^2 = 2.1897 \Rightarrow (\Delta a)^2 = (30 - 4 \cdot 0.5^2)^{-1}(2.1897/\{4-2\}) = 0.0378$
$\Rightarrow \Delta a = 0.1943 \approx 0.2$
and $(\Delta b)^2 = (1/4 + 0.5^2/\{30 - 4 \cdot 0.5^2\})(2.1897/\{4-2\}) = 0.2831$
$\Rightarrow \Delta b = 0.5321 \approx 0.6$ (rounded upward)
(2 points total)

If substitute answers $a = -2$ and $b = 3$ are used, then $r = (-3, -1, -1, 2)$ and $\sum r_i^2 = 15$ so $\Rightarrow (\Delta a)^2 = (30 - 4 \cdot 0.5^2)^{-1}(15/\{4-2\}) = 0.2586$
$\Rightarrow \Delta a = 0.5085 \approx 0.6$ (rounded upward)
and $(\Delta b)^2 = (1/4 + 0.5^2/\{30 - 4 \cdot 0.5^2\})(15/\{4-2\}) = 1.9397$
$\Rightarrow \Delta b = 1.3927 \approx 1.4$

c) $x = 1 \Rightarrow y = ax + b = a + b = 0.5517 \approx 0.6$ *(1 point for correct value, only 0.5 pt if rounded value is not given)*

If substitute answers $a = -2$ and $b = 3$ are used, then $a + b = 1$

Substitution $z = ax$ or method of partial derivatives yields:

$$(\Delta y)^2 = (a\Delta x)^2 + (x\Delta a)^2 + (\Delta b)^2 = (-1.3966 \cdot 0)^2 + (1 \cdot 0.1943)^2 + 0.5321^2 = 0.3209$$

$\Rightarrow \Delta y = 0.5665 \approx 0.6$. Note: $\Delta x = 0$. *(2 points for correct formula, 1 point for correct value including correct number of only 1 significant digit)*
(4 points total)

If substitute answers $a = -2$ and $b = 3$ are used, together with $\Delta a = 0.6$ and $\Delta b = 1.4$ from part b), then $(\Delta y)^2 = (-2 \cdot 0)^2 + (1 \cdot 0.6)^2 + 1.4^2 = 2.32 \Rightarrow \Delta y = 1.5232 \approx 1.6$.

If substitute answers $a = -2, b = 3, \Delta a = 0.9$ and $\Delta b = 1.1$ are used, then $(\Delta y)^2 = (-2 \cdot 0)^2 + (1 \cdot 0.9)^2 + 1.1^2 = 2.02 \Rightarrow \Delta y = 1.4213 \approx 1.5$.

d) $\chi^2_{obs} = \sum\{r_i/(\Delta y_i)\}^2 = (-0.1379/1)^2 + (0.6552/2)^2 + (-1.1552/2)^2 + (0.6379/1)^2 = 0.8669$. *(3 points total: give 0.5 pt for knowing formula, -1 pt for forgetting squares in formula, no points if summation is forgotten, -1 pt if Δy = constant is used, -1.5 pt if mistake in calculation)*
(3 points total)

If substitute answers $a = -2$ and $b = 3$ are used, then $r = (-3, -1, -1, 2)$ so $\chi^2_{obs} = (-3/1)^2 + (-1/2)^2 + (-1/2)^2 + (2/1)^2 = 13.5$.

e) 2 parameters a and b are fitted, so $\nu = N - 2 = 2$. *(1 point)*
10% level $\Rightarrow \chi^2_{table} = 0.211$, 90% level $\Rightarrow \chi^2_{table} = 4.605$. χ^2_{obs} is in between those two limits, so the linear fit is acceptable. *(1 point)*
(2 points total)

If substitute answers $a = -2$ and $b = 3$ are used, then $\chi^2_{obs} = 13.5$, which is larger than 4.605 (90% level), so a linear fit is not acceptable in this case.